Ecography

E6370
Rhodes, J. R. and Jonzén, N. 2011. Monitoring temporal trends in spatially structured populations: how should sampling effort be allocated between space and time? – Ecography 34: xxx–xxx.

Supplementary material

Appendix 1

Derivation of the variance-covariance matrix of the model residuals

Base case

On a logarithmic scale, the Gompertz model defined in eq. 1 can be written as

$$\ln N_{i,t} = \ln N_{i,t-1} - 0.5\sigma^2 - \gamma\varepsilon_{i,t-1} + u_{i,t}, \tag{A1}$$

where $\varepsilon_{i,t-1} = \ln N_{i,t-1} - \ln K_{i,t-1}$. Now, since $\ln N_{i,t-1} = \ln K_{i,t-1} + \varepsilon_{i,t-1}$, we can write eq. A1 as

$$\begin{aligned}\ln N_{i,t} &= \ln K_{i,t-1} - 0.5\sigma^2 + \varepsilon_{i,t-1} - \gamma\varepsilon_{i,t-1} + u_{i,t} \\ &= \ln K_{i,t-1} - 0.5\sigma^2 + (1-\gamma)\varepsilon_{i,t-1} + u_{i,t}\end{aligned}, \tag{A2}$$

and because we assume that $\ln K_{i,t}$ changes deterministically at a rate *r*, we can write

$$\ln K_{i,t-1} = \ln K_{i,0} + r(t-1), \tag{A3}$$

for $t \geq 1$. Substituting eq. A3 into eq. A2 and using the fact that $\varepsilon_{i,t} = (1-\gamma)\varepsilon_{i,t-1} + u_{i,t}$, we obtain

$$\ln N_{i,t} = \ln K_{i,0} - r - 0.5\sigma^2 + rt + \varepsilon_{i,t}, \tag{A4}$$

where $\varepsilon_{i,t} = (1-\gamma)\varepsilon_{i,t-1} + u_{i,t}$. Equation A4 is equivalent to eq. 2.

Now we can derive the correlation structure of the residuals, $\varepsilon_{i,t}$, for this model. First,

$$\begin{aligned}\mathrm{Var}(\varepsilon_{i,t}) &= \mathrm{Var}((1-\gamma)\varepsilon_{i,t-1} + u_{i,t}) \\ &= (1-\gamma)^2\,\mathrm{Var}(\varepsilon_{i,t-1}) + \mathrm{Var}(u_{i,t})\end{aligned}$$

and, since $\mathrm{Var}(\varepsilon_{i,t}) = \mathrm{Var}(\varepsilon_{i,t-1})$, then

$$\mathrm{Var}(\varepsilon_{i,t}) = \frac{\mathrm{Var}(u_{i,t})}{1-(1-\gamma)^2}$$

and so

$$\mathrm{Var}(\varepsilon_{i,t}) = \frac{\sigma^2}{1-(1-\gamma)^2}, \tag{A5}$$

where $\sigma^2 = \mathrm{Var}(u_{i,t})$. Second,

$$
\begin{aligned}
\mathrm{Cov}(\varepsilon_{i,t},\varepsilon_{j,s}) &= \mathrm{Cov}((1-\gamma)\varepsilon_{i,t-1}+u_{i,t},(1-\gamma)\varepsilon_{j,s-1}+u_{j,s})\\
&= \mathrm{Cov}((1-\gamma)\varepsilon_{i,t-1},(1-\gamma)\varepsilon_{j,s-1})+\mathrm{Cov}((1-\gamma)\varepsilon_{i,t-1},u_{j,s})\\
&\quad +\mathrm{Cov}(u_{i,t},(1-\gamma)\varepsilon_{j,s-1})+\mathrm{Cov}(u_{i,t},u_{j,s})\\
&= (1-\gamma)^2\mathrm{Cov}(\varepsilon_{i,t-1},\varepsilon_{j,s-1})+(1-\gamma)\mathrm{Cov}(\varepsilon_{i,t-1},u_{j,s})\\
&\quad +(1-\gamma)\mathrm{Cov}(u_{i,t},\varepsilon_{j,s-1})+\mathrm{Cov}(u_{i,t},u_{j,s})
\end{aligned}
$$

and, since $\mathrm{Cov}(\varepsilon_{i,t},\varepsilon_{j,s})=\mathrm{Cov}(\varepsilon_{i,t-1},\varepsilon_{j,s-1})$, then

$$
\mathrm{Cov}(\varepsilon_{i,t},\varepsilon_{j,s})=\frac{(1-\gamma)\mathrm{Cov}(\varepsilon_{i,t-1},u_{j,s})+(1-\gamma)\mathrm{Cov}(\varepsilon_{j,s-1},u_{i,t})+\mathrm{Cov}(u_{i,t},u_{j,s})}{1-(1-\gamma)^2}. \tag{A6}
$$

When $t > s$, then

$$
\begin{aligned}
(1-\gamma)\mathrm{Cov}(\varepsilon_{i,t-1},u_{j,s}) &= (1-\gamma)^{|t-s|}\rho^{d_{i,j}}\sigma^2\\
(1-\gamma)\mathrm{Cov}(\varepsilon_{j,s-1},u_{i,t}) &= 0\\
\mathrm{Cov}(u_{i,t},u_{j,s}) &= 0
\end{aligned}
$$

and so, from eq. A6

$$
\mathrm{Cov}(\varepsilon_{i,t},\varepsilon_{j,s})=\frac{(1-\gamma)^{|t-s|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2}.
$$

When $t < s$, then

$$
\begin{aligned}
(1-\gamma)\mathrm{Cov}(\varepsilon_{i,t-1},u_{j,s}) &= 0\\
(1-\gamma)\mathrm{Cov}(\varepsilon_{j,s-1},u_{i,t}) &= (1-\gamma)^{|t-s|}\rho^{d_{i,j}}\sigma^2\\
\mathrm{Cov}(u_{i,t},u_{j,s}) &= 0
\end{aligned}
$$

and so, from eq. A6

$$
\mathrm{Cov}(\varepsilon_{i,t},\varepsilon_{j,s})=\frac{(1-\gamma)^{|t-s|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2}.
$$

When $t = s$ then

$$
\begin{aligned}
(1-\gamma)\mathrm{Cov}(\varepsilon_{i,t-1},u_{j,s}) &= 0\\
(1-\gamma)\mathrm{Cov}(\varepsilon_{j,s-1},u_{i,t}) &= 0\\
\mathrm{Cov}(u_{i,t},u_{j,s}) &= \rho^{d_{i,j}}\sigma^2
\end{aligned}
$$

and so, from eq. 6

$$
\mathrm{Cov}(\varepsilon_{i,t},\varepsilon_{j,s})=\frac{(1-\gamma)^{|t-s|}\rho^{d_{i,t}}\sigma^2}{1-(1-\gamma)^2}.
$$

In general, therefore

$$\mathrm{Cov}(\varepsilon_{i,t}, \varepsilon_{j,s}) = \frac{(1-\gamma)^{|t-s|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \tag{A7}$$

and

$$\mathrm{Corr}(\varepsilon_{i,t}, \varepsilon_{j,s}) = \frac{(1-\gamma)^{|t-s|}\rho^{d_{i,j}}}{1-(1-\gamma)^2}. \tag{A8}$$

If we sampled S subpopulations, each at T points in time, with no observation error, then, based on eq. A5 and eq. A7, the variance-covariance matrix of the residuals, $\varepsilon_{i,t}$, is

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{P}_{1,S} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \cdots & \mathbf{P}_{2,S} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{P}_{S,1} & \mathbf{P}_{S,2} & \cdots & \mathbf{P}_{S,S} \end{bmatrix}, \tag{A9}$$

where

$$\mathbf{P}_{i,j} = \begin{bmatrix} \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \end{bmatrix}.$$

Here, each sub-matrix, $\mathbf{P}_{i,j}$, represents the correlation structure among the residuals for surveyed subpopulations i and j among points in time.

Observation error

If we now introduce normally distributed observation errors, with variance, σ^2_{obs}, then, on a logarithmic scale, the observed abundances are

$$\ln \widetilde{N}_{i,t} = \ln N_{i,t} - 0.5\sigma^2_{obs} + v_{i,t}$$

where $v_{i,t}$ is a normally distributed random variable with mean zero and variance σ^2_{obs}. Hence, from eq. A4

$$\ln \widetilde{N}_{i,t} = \ln K_{i,0} - r - 0.5\sigma^2 - 0.5\sigma^2_{obs} + rt + \varepsilon_{i,t} + v_{i,t}, \tag{A10}$$

Which is equivalent to the model in eq. A4, but with an intercept equal to $\ln K_{i,0} - r - 0.5\sigma^2 - 0.5\sigma_{obs}^2$ and residuals equal to $\varepsilon_{i,t} + v_{i,t}$. If we set $\delta_{i,t} = \varepsilon_{i,t} + v_{i,t}$, then $\mathrm{Var}(\delta_{i,t}) = \mathrm{Var}(\varepsilon_{i,t}) + \mathrm{Var}(v_{i,t})$ and, because the $v_{i,t}$ are independent $\mathrm{Cov}(\delta_{i,t}, \delta_{j,s}) = \mathrm{Cov}(\varepsilon_{i,t}, \varepsilon_{j,s})$. Therefore, the variance-covariance matrix for the residuals, $\delta_{i,t}$, of the model with observation errors is

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{P}_{1,S} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \cdots & \mathbf{P}_{2,S} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{P}_{S,1} & \mathbf{P}_{S,2} & \cdots & \mathbf{P}_{S,S} \end{bmatrix}, \qquad \text{(A11)}$$

where, when $i = j$

$$\mathbf{P}_{i,j} = \begin{bmatrix} \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + \sigma_{obs}^2 & \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + \sigma_{obs}^2 & \cdots & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + \sigma_{obs}^2 \end{bmatrix}$$

and, when $i \neq j$

$$\mathbf{P}_{i,j} = \begin{bmatrix} \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \end{bmatrix}.$$

Variation in temporal trends among sub-populations

Now consider the case where each sub-population has a different deterministic trend, r_i, and that these trends are normally distributed with mean r and variance σ_{trend}^2. We will now ignore observation errors, but the extension to include observation errors is straightforward following the rationale described above. With variation in the trend among sub-populations, the model described in eq. A4 becomes

$$\ln N_{i,t} = \ln K_{i,0} - (r + \eta_i) - 0.5\sigma^2 + (r + \eta_i)t - 0.5\sigma_{trend}^2 + \varepsilon_{i,t},$$

where η_i is a normally distributed random variable with mean 0 and variance σ^2_{trend}. We can then write this model as

$$\ln N_{i,t} = \ln K_{i,0} - r - 0.5\sigma^2 - 0.5\sigma^2_{trend} + rt + \omega_{i,t}, \tag{A15}$$

where $\omega_{i,t} = (t-1)\eta_i + \varepsilon_{i,t}$.

Following this, the variance of the residuals, $\omega_{i,t}$, is

$$\mathrm{Var}(\omega_{i,t}) = (t-1)^2 \mathrm{Var}(\eta_i) + \mathrm{Var}(\varepsilon_{i,t}).$$

When $i = j$ the covariance among residuals, $\omega_{i,t}$ and $\omega_{j,s}$, is

$$\begin{aligned}\mathrm{Cov}(\omega_{i,t}, \omega_{j,s}) &= (t-1)(\mathrm{s}-1)\mathrm{Var}(\eta_\mathrm{i}) + \mathrm{Cov}(\varepsilon_{i,t}, \varepsilon_{j,s}) \\ &= (t-1)(\mathrm{s}-1)\sigma^2_{trend} + \mathrm{Cov}(\varepsilon_{i,t}, \varepsilon_{j,s})\end{aligned}.$$

When $i \neq j$, the covariance among residuals, $\omega_{i,t}$ and $\omega_{j,s}$, is

$$\mathrm{Cov}(\omega_{i,t}, \omega_{j,s}) = \mathrm{Cov}(\varepsilon_{i,t}, \varepsilon_{j,s}),$$

because $\mathrm{Cov}(\eta_i, \eta_j) = 0$, $\mathrm{Cov}(\eta_i, \varepsilon_{j,s}) = 0$ and $\mathrm{Cov}(\eta_j, \varepsilon_{i,t}) = 0$.

Therefore, the variance-covariance matrix of the residuals when there is variation among sub-population trends is

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \dots & \mathbf{P}_{1,S} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \dots & \mathbf{P}_{2,S} \\ \dots & \dots & \dots & \dots \\ \mathbf{P}_{S,1} & \mathbf{P}_{S,2} & \dots & \mathbf{P}_{S,S} \end{bmatrix}, \tag{A16}$$

where, when $i = j$

$$\mathbf{P}_{i,j} = \begin{bmatrix} \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \dots & \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\ \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + \sigma^2_{trend} & \dots & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + (T-1)\sigma^2_{trend} \\ \dots & \dots & \dots & \dots \\ \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + (T-1)\sigma^2_{trend} & \dots & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} + \left((T-1)^2\sigma^2_{trend}\right) \end{bmatrix}$$

and, when $i \neq j$

$$
\mathbf{P}_{i,j} = \begin{bmatrix}
\frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\
\frac{(1-\gamma)\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{(1-\gamma)^{|T-1|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \frac{(1-\gamma)^{|T-2|}\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2} & \cdots & \frac{\rho^{d_{i,j}}\sigma^2}{1-(1-\gamma)^2}
\end{bmatrix}.
$$

Matlab code

The two Matlab .m files embedded below contain the functions 'getsurveyo' and 'getsurveyv'. The function 'getsurveyo' returns the variance-covariance matrix for a survey with observation error. The function 'getsurveyv' returns the variance-covariance matrix for a survey with variation in trends among sub-populations.

Download <getsurveyo.m>
<getsurveyv.m>